Mathematics

Paper-II

Time Allowed: Three Hours **Maximum Marks: 300**

Note: Candidate should answer questions No. 1 and 5 which are compulsory and any ***three*** *of the remaining questions, selecting at least* ***one*** *from each section.*

SECTION – A

1. Answer any **five** of the following: 12×5=60

(a) Show that the set $\{1, 3, 4, 5, 9\}$ is an abelian group under multiplication modulo 11.

(b) Prove that a field has no proper ideals.

(c) Test the convergence and absolute convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p},\ p > 0$.

(d) Show that the function f defined on [0, 1] by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

is bounded but not R-integrable.

(e) State and prove Cauchy integral formula.

(f) Using residue theorem, evaluate $\int_C \frac{e^z dz}{z(z-1)^2}$

where c is circle $|z| = 2$.

(g) Solve the following LPP by graphical method:

Minimize $z = 8x + 5y$

subject to the constraints

$2x + y \geq 8,\ 6x + y \geq 12,\ x + 3y \geq 9,\ x, y \geq 0$

2. Answer all the **five** parts: 12×5=60

(a) Prove that the intersection of any family of subgroups of a group is a subgroup.

(b) If G be a group of positive real numbers under multiplication and $f: G \to G$ be such that $f(x) = x^2$, then show that f is an automorphism.

(c) Write the permutation $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 1 & 4 & 8 & 6 & 9 & 7 & 5 \end{pmatrix}$ as the product of disjoint cycles.

(d) Show that none of the following sets form an integral domain:

(i) set of even integers.

(ii) set of 3 × 3 matrices.

(e) Solve the following cost-minimizing problem:

Machines	Jobs				
	I	**II**	**III**	**IV**	**V**
A	11	10	18	5	9
B	14	13	12	19	6
C	5	3	4	2	4
D	15	18	17	9	12
E	10	11	19	6	14

3. Answer all the **five** parts: $12\times5=60$

(a) What are supremum and infimum of a set? Do they always belong to the set? Show by examples.

(b) Show that if a function f is uniformly continuous on $[a, b]$, then it is continuous on $[a, b]$, but the converse is not true.

(c) Prove that every Cauchy's sequence is bounded. Show by an example that the converse is not true.

(d) Test the following improper integral for convergence:

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\, dx$$

(e) Determine the initial basic feasible solution using Vogel's method for the following transportation problem:

Source	**Destination**				**Supply**
	1	2	3	4	
1	21	16	15	13	11
2	17	18	14	23	13
3	32	27	18	41	19
Demand	6	10	12	15	43

4. Answer all the **five** parts: $12\times5=60$

(a) Determine the analytic function $f(z)=u+iv$ of which the real part is $u=e^x(x\cos y-y\sin y)$.

(b) Obtain the Taylor and Laurent's series which represents the function $\frac{1}{z^2-3z+2}$ in the regions (i) $0<|z|<1$, (ii) $1<|z|<2$, (iii) $|z|>2$.

(c) Prove that $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \frac{\pi}{6}$.

(d) Find the bilinear transformation which maps the points $z = 0, -1, \infty$ into the points $w = -1, -2-i, i$ respectively.

(e) Apply simplex method to solve the LPP

Maximize $z = 30x_1 + 23x_2 + 29x_3$, subject to $6x_1 + 5x_2 + 3x_3 \leq 26$, $4x_1 + 2x_2 + 6x_3 \leq 7$, $x_1, x_2, x_3 \geq 0$.

SECTION – B

5. Answer any **five** of the following: 12×5=60

(a) Find the differential equation of all spheres of fixed radius having centre in xy-plane.

(b) Solve the differential equation:

$$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + \frac{\partial^3 z}{\partial y^3} = e^{x+2y}$$

(c) Evaluate $\sqrt{29}$ to five places by Newton-Raphson method.

(d) Solve $\int_0^{12} \frac{dx}{1+x^2}$ using Simpson's $\frac{1}{3}$ rule taking $n = 6$.

(e) Write a BASIC program to compute the sum of two matrices.

(f) (i) What is a bit? What is a byte? What is the difference between a byte and a word of memory?

(ii) Describe the precedence and the associativity for the bitwise shift operator.

(iii) Convert $BE38_{16}$ to octal and decimal.

(g) Describe the following:

(i) Generalized coordinates.

(ii) Holonomic and non-holonomic systems with examples.

6. Answer all the **five** parts: 12×5=60

(a) Form a PDE by eliminating the arbitrary function ϕ from $\phi(x + y + z, x^2 + y^2 - z^2) = 0$. What is the order of this PDE?

(b) Solve the partial differential equation (PDE)

$$x(y-z)p+y(z-x)q=z(x-y)$$

(c) Find the complete integral of $p^2y(1+x^2)=qx^2$

(d) Find the equation of surface satisfying $4yzp+q+2y=0$ and passing through $y^2+z^2=1, x+z=2$.

(e) Show that the general solution of the PDE $\frac{\partial^2 z}{\partial x^2}=\frac{1}{C^2}\frac{\partial^2 z}{\partial t^2}$ is of the form

$z(x,t)=F(x+Ct)+G(x-Ct)$ where F and G are arbitrary functions.

7. Answer all the **five** parts: 12×5=60

(a) Determine a root of the equation $x^3-x^2-5=0$ using Regula-Falsi method.

(b) Solve the following equations by Gauss-Jordan method:

$$x+3y+2z=17;\ x+2y+3z=16;\ 2x-y+4z=13$$

(c) Using Lagrange's interpolation formula, find the value of y corresponding to $x = 10$ from the following data:

x	5	6	9	11
y	380	−2	196	508

(d) Evaluateas $\int_{-1}^{1}\left(1-x^2\right)^{\frac{3}{2}} dx$ accurately as possible using Gauss-Legendre quadrature.

(e) Use the Runge-Kutta fourth-order method to solve $\frac{dy}{dx}=\frac{y-x}{y+x}$ with $y(0)=1$ at $x=0.2$.

8. Answer all the **five** parts: 12×5=60

(a) Derive Hamiltonian and equation of motion for a simple pendulum.

(b) Show that the moment of inertia of a uniform rectangular mass M and sides $2a$ and $2b$ about a diagonal is $\frac{2Ma^2b^2}{3\left(a^2+b^2\right)}$.

(c) Derive equation of continuity by Euler's method.

(d) Suppose $\vec{v}=(x-4y)\hat{i}+(4x-y)\hat{j}$ represents a velocity field of an incompressible and irrotational flow. Find the stream function of the flow.

(e) Show that the vorticity vector $\vec{\zeta}$ of an incompressible viscous fluid moving under no external forces satisfies the differential equation

$\frac{d\vec{\zeta}}{dt}=\left(\vec{\zeta}\cdot\nabla\right)\vec{q}+\mu\nabla^2\vec{\zeta}$, where μ is the coefficient of viscosity.