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MATHEMATICS

PAPER—I

Time Allowed : Three hours *Maximum Marks* : 300

The figures in the margin indicate full marks for the questions

Question Nos. **1** and **5** are compulsory. Candidates should answer **three** questions from the rest, selecting at least **one** from each Section

SECTION—A

1. Answer any *five* of the following : 12×5=60

(a) Show that the vectors (1, 3, 2), (1, −7, −8) and (2, 1, −1) in $\mathbb{R}^3$ over $\mathbb{R}$ are linearly dependent.

(b) Prove that the vector space $\mathbb{R}^2$ over reals $\mathbb{R}$ is the direct sum of the subspaces $W_1 = \langle(2, 3)\rangle$ and $W_2 = \langle(3, 2)\rangle$.

(c) Show that $\sin x(1+\cos x)$ is a maximum at $x = \frac{\pi}{3}$ and neither maximum nor minimum at $x = \pi$.

(d) Show that

$$\int_0^1\left\{\int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dy\right\}dx = \int_0^1\left\{\int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dx\right\}dy$$

(e) Find the centre and radius of the circle $x^2+y^2+z^2-2y-4z=11$, $x+2y+2z=15$.

(f) Find the magnitude and the equations of the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$.

2. Answer the following five questions : 12×5=60

(a) Show that {(1, 1, 0), (1, 0, 1), (0, 1, 1)} form a basis for $\mathbb{R}^3$ over $\mathbb{R}$.

(b) Show that the matrix

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

is an orthogonal matrix.

(c) Reduce the following matrix to its echelon form and find its rank :

$$A = \begin{pmatrix} 2 & 3 & 1 & 2 & 0 \\ 0 & 3 & -1 & 2 & 1 \\ 1 & -3 & 2 & 4 & 3 \\ 2 & 3 & 0 & 3 & 0 \end{pmatrix}$$

(d) Verify the Cayley-Hamilton theorem for the matrix

$$A = \begin{pmatrix} -3 & 5 & 1 \\ 2 & 0 & -1 \\ 1 & -2 & 3 \end{pmatrix}$$

(e) Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

3. Answer the following five questions : 12×5=60

(a) If

$$f(x,\ y) = \begin{cases} \dfrac{x^3 - y^3}{x^3 + y^3}, & (x,\ y) \neq (0,\ 0) \\ 0, & (x,\ y) = (0,\ 0) \end{cases}$$

show that

(i) f is continuous at (0, 0)

(ii) $f_x(0,\ 0) \neq f_y(0,\ 0)$

(iii) f is not differentiable at (0, 0)

(b) If $u = r\cos\theta$ and $v = r\sin\theta$, then show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0 \text{ becomes } \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = 0$$

(c) Show that

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2} \quad \text{for } b > a > 0$$

(d) Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the chord $y = mx$.

(e) Show that $2^n \Gamma(n+\frac{1}{2}) = 1.3.5. \cdots (2n-1)\sqrt{\pi}$, where n is a positive integer.

4. Answer the following five questions : 12×5=60

(a) A variable plane is at a distance p from the origin and meets the axes at A, B, C. Find the locus of the centroid of the tetrahedron $OABC$.

(b) Find the equation of the plane through the origin containing the line

$$\frac{x-1}{5} = \frac{y-2}{4} = \frac{z-3}{5}$$

(c) Find the equation of the sphere through the points (0, 0, 0), (0, 1, −1), (−1, 2, 0) and (1, 2, 3).

(d) Find the equation of the right circular cone generated when the straight line $2y + 3z = 6$, $x = 0$ revolves about z-axis.

(e) Find the equation of the right circular cylinder if the radius of a normal section of the cylinder is 2 units and axis lies along the straight line

$$\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{5}$$

SECTION—B

5. Answer any *five* of the following : 12×5=60

(a) Solve

$$\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$$

(b) Solve

$$(D^2 - 1)y = x\sin 3x + \cos x$$

(c) A particle moves along a straight line, its distance x from a fixed point O on the line is $k\sqrt{\frac{c-x}{x}}$. Prove that the acceleration is directed towards O and is inversely proportional to the square of its distance from O.

(d) $ABCDEF$ is a regular hexagon of side a and forces represented in magnitudes and directions by $\overrightarrow{AB}$, $2\,\overrightarrow{AC}$, $3\,\overrightarrow{AD}$, $4\,\overrightarrow{AE}$, $5\,\overrightarrow{AF}$ act at A. Show that the magnitude of their resultant is $\sqrt{351}\,a$.

(e) Prove that

(i) $\text{div}\,\text{curl}\,\mathbf{F} = \nabla \cdot \nabla \times \mathbf{F} = 0$

(ii) $\text{curl}\,\text{curl}\,\mathbf{F} = \text{grad}\,\text{div}\,\mathbf{F} - \nabla^2 \mathbf{F}$

(f) Prove that a curve be a helix if and only if its curvature and torsion are in a constant ratio.

6. Answer the following five questions : 12×5=60

(a) Solve

$$xy(1 + xy^2)\frac{dy}{dx} = 1$$

(b) Solve

$$\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$$

(c) Find the general and singular solutions of the equation $y = px - \sqrt{1 + p^2}$.

(d) Using the method of variation of parameters, solve

$$\frac{d^2y}{dx^2} + 4y = \tan 2x$$

(e) Solve

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$$

7. Answer the following five questions : 12×5=60

(a) A particle starts from rest and moves along a straight line with uniform acceleration f. At the end of time t, the acceleration becomes $2f$; at the end of time $2t$, it becomes $3f$ and so on. Show that the velocity at the end of time nt is $\frac{1}{2}n(n+1)ft$.

(b) A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is k (velocity). Show that the velocity v and the distance s after time t are given by $v = Ve^{-kt}$ and $s = \frac{V}{k}(1 - e^{-kt})$.

(c) Forces 3, 2, 4, 5 kg wt act respectively along the sides AB, BC, CD, DA of a square $ABCD$. Find the magnitude of their resultant and the point where its line of action meets AB.

(d) Show that the system of coplanar forces acting on a rigid body is reducible to a single force acting at an arbitrary chosen point in the plane together with a couple in the plane. When will the system be in equilibrium?

(e) Discuss *(i)* equilibrium of fluids under a system of forces and *(ii)* centre of pressure.

8. Answer the following five questions : 12×5=60

(a) Find grad f for $f(\mathbf{r}) = 3x^2 + 2y^2 + z^2$ at the point (1, 2, 3). Hence calculate the directional derivative of $f(\mathbf{r})$ at (1, 2, 3) in the direction of the unit vector $\frac{1}{3}(2, 2, 1)$.

(b) If $\mathbf{r}$ is the usual position vector $\mathbf{r} = (x, y, z)$, show that

(i) $\text{div grad}\left(\frac{1}{r}\right) = 0$

(ii) $\text{curl}\left[\mathbf{k} \times \text{grad}\left(\frac{1}{r}\right)\right] + \text{grad}\left[\mathbf{k}.\text{grad}\left(\frac{1}{r}\right)\right] = 0$

(c) If $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + (at\tan\alpha)\mathbf{k}$, find the value of

(i) $\left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right|$

(ii) $\left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}\right]$

(d) Find the curvature and torsion of the curve $x = a\cos t$, $y = a\sin t$, $z = bt$.

(e) State and prove Serret-Frenet formulae for a space curve.

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