Mathematics

Paper-II

Time Allowed: Three hours **Maximum Marks: 300**

The figures in the margin indicate full marks for the questions :

Note: Candidates should answer Question nos. **1** and **5** which are compulsory and any **three** from the rest selecting at least **one** from each section.

SECTION-A

1. Answer any **five** from the following : 12×5=60

(a) Show that if every element of the group G is its own inverse, then G is abelian.

(b) Prove that every finite Integral domain is a field.

(c) Show that the sequence $\{S_n\}$ where

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+n}$$

is convergent.

(d) Show that the funcation defined by $f(x) = x^2$ is uniformly continuous on the interval $[-1, 1]$.

(e) Find whether the function

$f(z) = z^3 + z$

is analytic.

(f) Find the residues of the function

$$f(z) = \frac{2z}{(z+4)(z-1)^2}$$

at its poles.

(g) Find basic feasible solution of the Linear Programming Problem.

Max. $z = 3x_1 + 5x_2$

such that

$x_1 + 2x_2 + x_3 = 6$

$4x_1 + 3x_2 + x_4 = 12$

$x_1, x_2, x_3, x_4 \geq 0$

2. Answer all the following **five** parts : 12×5=60

(a) Prove that the set G = {0, 1, 2, 3, 4} is a finite abelian group of order 5 under addition modulo 5 as the composition in G

(b) Show that the additive group of integers

$G = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

is isomorphic to the additive group

$G = \{..., -3m, -2m, -1m, 0, 1m, 2m, 3m, ...\}$

where $m \neq 0$ is any constant integer.

(c) If M and N are normal subgroups of a group G, then show that $M \cap N$ is also a normal subgroup of G

(d) Show that a field has no zero divisors.

(e) If a, b, c, d are elements of a ring R, then evaluate $(a + b)(c + d)$.

3. Answer all the **five** parts : 12×5=60

(a) If the sequences $\{s_n\}$ converges to L, prove that the sequence $\{(-1)^n s_n\}$ oscillates.

(b) Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on (0 ,1) but it is uniformly continuous on (a, ∞) where $a > 0$.

(c) Examine the convergence of the series

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} \text{ for } 0 < x < 1$$

(d) Show that each constant function $f(x) = c$ is Riemann integrable on any interval $[a, b]$.

(e) Find the maxima and minima of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

4. Solve all the **four** parts : 15×4=60

(a) Derive C-R equations for an analytic function.

(b) Find the Laurent series for the function

$$f(z) = \frac{1}{1 - z^2} \text{ with centre at } z = 1.$$

(c) In conformal mapping $f : G \to C$, where $f(z)$ is analytic. Then prove that f preserves angle at each point z_0 of G where $f'(z_0) \neq 0$

(d) Using contour integration evaluate

$$\int_{-\infty}^{\infty} \frac{x^2\, dx}{(x^2 + 1)(x^2 + 4)}$$

SECTION-B

5. Solve any **five** of the following : 12×5=60

(a) From $2z=(ax+y)^2+b$, from the partial differential equation.

(b) Solve the partial differential equation

$$(mz-ny)\frac{\partial z}{\partial x}+(nx-lz)\frac{\partial z}{\partial x} = ly-mx$$

(c) Solve the following equation by false position (Regula-falsi) method

$x^3-2x-5=0$

(d) The population of a town in the decennial census was given below:

Year x	1901	1911	1921	1931	1941
Population y in thousands	46	66	81	91	101

Find the population for the year 1905.

(e) Evaluate the following integral by Simpson's one-third rule for $h=0.125$

$$\int_0^1 \frac{1}{1+x}dx$$

(f) Write a program in Basic to write the following numbers in ascending order

7, 2, 9, 3, 4, 6, 5, 8, 1

(g) Derive the equation of continuity in Cartesian co-ordinates.

6. Solve all the **five** parts : 12×5=60

(a) Construct partial differential equation from

$f(x^2+y^2, z-xy)=0$

(b) Find the solution of

$$\sqrt{x}\frac{\partial z}{\partial x}+\sqrt{y}\frac{\partial z}{\partial y} = \sqrt{z}$$

(c) Solve by Charpit's method

$$\frac{\partial z}{\partial x} = \left(z+y\frac{\partial z}{\partial y}\right)^2$$

(d) Find the general solution of

$$\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x\partial y}+4\frac{\partial^2 z}{\partial y^2} = 0$$

(e) Construct the partial differential equation for a vibrating string fixed at the two ends.

7. Solve all the **five** parts : 12×5=60

(a) Find real roots of the equations

$x^2 - y^2 = 4,\ x^2 + y^2 = 16$

by Newton-Raphson method.

(b) Using Lagrange's Interpolation formula find the value of log 301 from the follow ing table

x	300	304	305	307
$\log x$	2.4771	2.4829	2.4843	2.4871

(c) Evaluate the integral $\int_0^1 x\, dx$ by Gauss qurature formula.

(d) Find the numerical solution of the differential equation

$$\frac{dy}{dx} = 1 + y^2 \quad \text{for } y(0) = 0$$

at $x = 0.2, 0.4, 0.6$ using Runge - Kutta method by taking h = 0.2

(e) (i) Explain the following in computer :

bits, bytes, words

(ii) Illustrates use of Algorithms and Flow charts in solving equations by bisec tion method.

8. Solve all the **four** parts : 15×4=60

(a) A mass of fluid is in motion so that the lines of motion lie on the surface of coaxial cylinders. Find the equation of continuity.

(b) Find the velocity potential and stream function for a liquid flowing past a fixed elliptic cylinder with veloicity v parallel to the major axis.

(c) Write Hamilton equations and explain.

(d) Find moment of inertia of a thin homogeneous rod of length l about its end point.

Mathematics

(Main Examination)

Paper-I

Time Allowed: Three hours **Maximum Marks: 300**

The figures in the margin indicate full marks for the questions.

Questions nos. **1** and **5** are compulsory. Candidates should answer any **three** questions from the rest, selecting at least **one** from each section.

SECTION-A

1. Answer any **five** of the following : 12×5=60

(a) Let V be a vector space over a field F. Suppose that

$S = \{x_1, x_2, \dots x_n\}$

is a subset of non-zero elements of V. Then show that S is linearly dependent if and only if there is a k such that $1 \le k < n$ and

$x_{k+1} \in [\{x_1, x_2, \dots, x_k\}]$

(b) Show that the matrix $\frac{1}{5}\begin{bmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{bmatrix}$ is unitary, while the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is orthogonal.

(c) Examine the following function for maximum and minimum :

$y = 2\sin x + \cos 2x$

in the interval $[0, 2\pi]$.

(d) Compute the area of a region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

by using integration.

(e) Evaluate the limit

$$\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\tan 3x}$$

(f) State and prove Taylor's theorem.

(g) Find equation of ellipse whose centre is at the origin and passes through the points (2, 2) and (3, 1).

2. Answer all the **five** parts : 12×5=60

(a) Prove that the four vectors $\alpha_1 = (1, 2, 3)$, $\alpha_2 = (1, 0, 0)$, $\alpha_3 = (0, 1, 0)$ and $\alpha_4 = (0, 0, 1)$ in $\mathbb{R}^3$ form a linearly dependent set.

(b) Find the rank of a matrix

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

(c) Find the minimal polynomial of the following matrix :

$$\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

(d) Find all the eigen values and corresponding eigen vectors of the matrix

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

(e) State and prove Cayley-Hamilton theorem

3. Answer all the **five** parts : 12×5=60

(a) Find the derivative of the function

$$y = \frac{(x+1)^2\sqrt{x-1}}{(x+4)^3\, e^x}$$

(b) State and prove mean-value theorem.

(c) Find the asymptotes of the curve

$$y = \frac{x^2 + 2x - 1}{x}$$

(d) Test the following function for maximum and minimum :

$$z = x^2 - xy + y^2 + 3x - 2y + 1$$

(e) Define Gamma and Beta functions and show that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

4. Answer all the **five** parts : 12×5=60

(a) Find the equation to the circle which passes through the points (1, 0), (0, – 6) and (3, 4).

(b) Find equations of the line through the point (1, 1, 1) which meet both the lines

$$\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-2}{4},\ x=2y=3z$$

(c) Find equations to the spheres which pass through the circle $x^2+y^2+z^2=5,\ x+2y+3z=3$ and touch the plane $4x+3y=15$.

(d) Find equation to the cone with vertex at the origin which pass through the curve given by

$x^2+y^2+z^2+2ax+b=0,\ lx+my+nz=p$

(e) Show that the plane $8x-6y-z=5$ touches the paraboloid $\frac{x^2}{2}-\frac{y^2}{3}=z$. Find the coordinates of the point of contact.

SECTION-B

5. Answer any **five** of the following : 12×5=60

(a) Solve the differential equation

$$\frac{dy}{dx}=-xy+x^3y^3$$

(b) Find complete solution of the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^{5x}$$

(c) Solve the differential equation

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

(d) Write the equation of motion of a particle moving along a straight line with known initial velocity and acceleration.

(e) One cubical box of one side equal to 'a' is placed on a fixed solid sphere such that the center of the lower face of the box is in touch with the highest point of the sphere. Find the minimum radius of the sphere for which the equilibrium is stable.

(f) Find the directional derivative of

$f(x, y, z)=xy^2+yz^3$ at (2, – 1, 1)

in the direction of the vector $i+2j+2k$

(g) Show that

curl $(\phi \text{ grad } \phi)=\bar{0}$

6. Solve all the **five** parts : 12×5=60

(a) Solve the equation

$$\frac{dy}{dx} = \frac{2y}{x+1} + (x+1)^3$$

(b) Find the general and singular solutions of the equation

$$y = x\frac{dy}{dx} + a\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-\frac{1}{2}}$$

(c) Solve the differential equation

$$(1 + xy)\, x\, dy + (1 - xy)\, y\, dx = 0$$

(d) Solve the differential equation

$$\frac{d^3y}{dx^3} + y = 3 + e^x + 5e^{2x}$$

(e) Solve the differential equation

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$$

7. Solve all the **four** parts : 15×4=60

(a) In a Simple Harmonic Motion is amplitude is 'a' and period 'T' then find the relation between distance x from the center and velocity v at that point.

(b) Find the velocity of projection and direction of a bullet which just crosses a wall at 50 metres away and 25 m high in horizontal direction.

(c) If the maximum and minimum velocities of planet moving round the sun are respectively 30 and 29.2 km/sec, then find the eccentricity of the orbit.

(d) State and explain Bernoulli's equation.

8. Solve all the **five** parts : 12×5=60

(a) Prove the derivative of a constant vector is zero.

(b) Find the angle between the tangents to the curve $\bar{r} = t^2 i + 2tj - t^3 k$ at the points $t = 1$ and $t = -1$.

(c) Show that curl curl $\bar{f} = 0$ for $\bar{f} = zi + xj + yk$

(d) Evaluate $\iint_S \left[(x^3 - yz)\, dydz + z\, dxdy - 2x^2 ydzdx\right]$

over the surface bounded by the planes $x = y = z = a$.

(e) Verify Stoke's theorem for $\bar{f} = yi + zj + xk$ where S is the upper half of $x^2 + y^2 + z^2 = 1$ and C is its boundary.